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Controlling flow turbulence with moving controllers

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Abstract. In this work, we consider how to improve the efficiency of controlling flow turbulence in two-dimensional Navier-Stokes equations. We suggest a control strategy which applies local feedback injections by moving controllers. In the moving frame, this strategy is equivalent to adding a gradient force term in the governing equation. It is shown that with the moving controllers, flow turbulence can be controlled more efficient than the usual pinning strategy with static controllers, provided that the number of controllers and the injection energy are the same. The physical mechanism underlying this higher control efficiency is heuristically analyzed. The advantages and difficulties of the proposed control strategy in practical applications are discussed.

PACS. 05.45.Gg Control of chaos, applications of chaos – 47.27.Rc Turbulence control – 05.45.Xt Synchronization; coupled oscillators

1 Introduction

For more than one century the physics of flow turbulence and the control of flow turbulence have attracted great interest of scientists and engineers due to its fascinating complexity and exceeding importance in practice [1–7]. After decades of extensive investigation in flow turbulence control, many passive and active methods for controlling flow turbulence have been successfully developed in engineering fields [8–16]. However, many difficulties and challenging problems still remain in this domain so far.

In the past decade, the topic of chaos control has attracted great interest in the field of nonlinear science due to its theoretical significance and potential in practical applications [17–21]. Recently, much attention has been paid to controlling high-dimensional spatiotemporal chaos [22–32]. For example, the control of "phase turbulence" and "defect turbulence" in excitable and oscillatory systems has been extensively studied [33–41]. It is generally believed that the control of flow turbulence could benefit from those strategies developed in controlling high-dimensional chaotic systems. However, controlling flow turbulence based on the understandings of nonlinear dynamics and chaos control is just at the very start. Recently, the global and local pinning control methods developed in spatiotemporal chaos control have been applied to control the flow turbulence described by incompressible Navier-Stokes equations (NSE) [42,43].

Control efficiency, i.e., how to achieve successful control as fast as possible with as few as possible number of controllers, is of particular importance in control problems, especially in practical applications. The main purpose of the present work is to enhance the efficiency of turbulence control with pinning strategy. It is found that by using local pinning strategy with moving controllers, flow turbulence can be controlled more efficiently, compared with the usual static pinning control. More importantly, this improvement of control efficiency is not at the cost of increasing the number of controllers and the injected energy. Numerical examples based on two-dimensional NSE are provided to illustrate the idea. The control efficiency in terms of convergence speed to the target with moving controllers is compared with that of the usual static pinning control. The mechanism underlying this high control efficiency is heuristically analyzed.

The paper is organized as follows. In Section 2, the dynamical model, the numerical scheme, and the pinning control method are introduced. Section 3 is devoted to the theoretical description of controlling flow turbulence with moving controllers. It is shown that the moving controller strategy is equivalent to control NSE in the presence of an

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additional gradient force. The detailed results of controlling flow turbulence with moving controllers are presented in Section 4. Brief conclusion is presented at the end of this paper.

2 Theoretical model

In this paper, we consider flow turbulence described by the following incompressible two-dimensional NSE [42,43]

$$\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u} = -\nabla p + \frac{1}{Re}\nabla^2\vec{u}, \tag{1a}$$

$$\nabla\cdot\vec{u} = 0, \tag{1b}$$

with $\vec{u} = (u, v)$, and $\vec{r} = (x, y)$. Here p is the pressure, and Re the Reynolds number. Space periodic boundary conditions $x + 2\pi = x$, $y + 2\pi = y$ are applied. Throughout this paper we keep $Re = 5000$, and the flow is thus in the regime of fully developed turbulence. For the numerical treatment, Fourier pseudospectral method, the Adams-Bashforth-Crank-Nicolson scheme [44], and the de-aliasing technique [45] are used in our simulations. Spatial discretization of a $2\pi \times 2\pi$ computational domain is performed in 256×256 grid points. In the computation, doubly periodic boundary condition is assumed. The validity of numerical results is confirmed by varying space and time steps. For the freely decaying two-dimensional flow turbulence, initial conditions are important for the flow dynamics. Usually, the initial conditions are assigned in Fourier space with specific energy spectrum such as

$$E(k, 0) \sim ke^{-(k/k_0)^2}, \tag{2}$$

where k is the wave number and k_0 a constant. In the current study, we use this initial energy spectrum with $k_0 = 5.0$. In Figure 1, the evolution of two-dimensional flow turbulence is shown, where the typical characteristics of two-dimensional flow turbulence, such as vortices forming, vortices collision and merging, and vortices diffusion are clearly seen. We use this turbulent dynamics as the reference for control.

The local pinning control of turbulence is to apply feedback signals

$$-\varepsilon'(\vec{u} - \vec{u}_T) \tag{3}$$

to some local regions of the space domain, where $\vec{u}_T$ is the target state. In the current work, two different targets $\vec{u}_{T1}$ and $\vec{u}_{T2}$ are considered. $\vec{u}_{T1}$ is a laminar flow

$$u_{T1} = 0.05, \quad v_{T1} = 0.0, \tag{4a}$$

and $\vec{u}_{T2}$ is a spatially periodic state, i.e.,

$$\begin{aligned} u_{T2} &= -\gamma\cos(x)\sin(y)e^{-2t/Re}, \\ v_{T2} &= \gamma\sin(x)\cos(y)e^{-2t/Re}. \end{aligned} \tag{4b}$$

with $\gamma = 0.1$. Both $\vec{u}_{T1}$ and $\vec{u}_{T2}$ are the exact solutions of NSE.

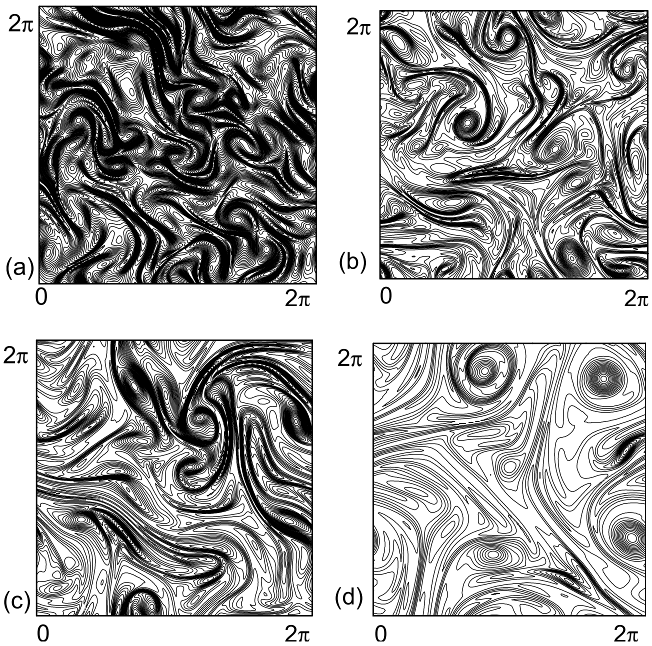

Fig. 1. Contour pictures of the evolution of the vorticity field of equation (1). $Re = 5000$. The spatial resolution is 256×256. The time step is $\Delta t = 0.001$. These parameters are used throughout the paper. The initial conditions are given by equation (2) with the initial energy $E(t = 0) = 0.01$. (a) $t = 5$; (b) $t = 30$; (c) $t = 60$; (d) $t = 100$.

In numerical simulation, the grid points where the control signals are added are called controllers. Specifically, with the pinning feedback control, equation (1a) becomes

$$\begin{aligned} \frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u} = -\nabla p + \frac{1}{Re}\nabla^2\vec{u} \\ -\sum_{i=1}^{M}\varepsilon'\delta(\vec{r} - \vec{r}_i)(\vec{u} - \vec{u}_T), \end{aligned} \tag{5}$$

where $\vec{r}_i$, $i = 1, 2, ..., M$, are the indices of controllers. In the present work, the controllers are uniformly distributed in the computational domain. The total number of controllers is

$$M = \frac{(256)^2}{I_x I_y}, \tag{6}$$

where I_x and I_y are the distances between two neighboring controllers in x and y directions, respectively. In the theoretical model of equation (5), the control forces with infinitely large amplitude are applied to a number of space points with zero area. In numerical simulations and experimental applications, the actual control forces are finite and they are applied in space grid points with nonzero area. Specifically, in our simulations we directly use a feedback force $\varepsilon(\vec{u} - \vec{u}_T)$ to each controlled grid where ε is scaled from ε' by a fact $\varepsilon = \frac{\varepsilon'}{|\Delta x||\Delta y|}$ with Δx and Δy being the

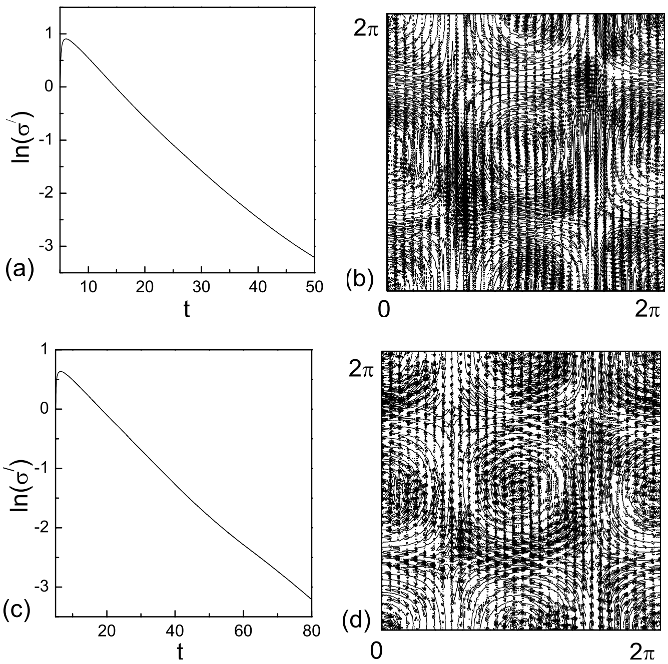

Fig. 2. Controlling flow turbulence with static controllers. The governing equation is equation (5). $\varepsilon = 5$. The pinning control is applied to the reference state (see Fig. 1) from $t = 5.0$. The target state is $\vec{u}_{T2}$. For (a) and (b), $I_x = 8$, $I_y = 4$. (a) $\ln(\sigma'(t))$ vs t. (b) vorticity contour at $t = 50$. For (c) and (d), $I_x = 8$, $I_y = 8$. (c) $\ln(\sigma'(t))$ vs. t. (d) vorticity contour at $t = 80$.

space discretization steps in x and y directions, respectively (note, we use $|\Delta x| = |\Delta y| = \frac{2\pi}{256}$ throughout the paper). It is obvious that $\varepsilon \longrightarrow \infty$ as $|\Delta x|, |\Delta y| \longrightarrow 0$. And this agrees with the delta function of equation (5) in the infinite resolution limit. In the following we will use ε instead of ε' as the control strength coefficient. For the pinning control applied in equation (5), the control efficiency is determined by the control strength ε, the total number of controllers M, and the space distribution of the controllers. In this paper, we fix $\varepsilon = 5$ and focus on the discussion on the control efficiency for different numbers of controllers (different M's).

In Figure 2, we present the control results with the target of equation (4b) with various numbers of controllers. It is observed that local pinning control can successfully suppress flow turbulence as long as the density of controllers is sufficiently large. Numerically, it is found that no satisfactory control is achieved when the proportion of the controllers to the total grids is smaller than 6%. This agrees with the results in reference [42].

3 Controlling flow turbulence with moving controllers

The central task of the present paper is to enhance the efficiency of controlling flow turbulence. Namely, we are interesting in how to enhance the speed with which the turbulence converges to the target, or to enlarge the parameter region within which the turbulence can be entrained. Intuitively, one may increase the number of controllers or increase the energy of the driving signals to achieve this purpose. Nevertheless, these methods are trivial in the sense that they rely on the injection energy and do not make use of the dynamical properties of the system to be controlled. Practically, it is highly desirable for us to improve the control efficiency without increasing the number of controllers or increasing the injection energy. Therefore, in this work, we are looking for nontrivial methods which can improve the control efficiency while keeping M and ε in equation (5) unchanged.

Here we propose a control strategy which uses moving controllers. The idea of moving controllers is as follows. Instead of injecting the feedback signals (3) into the fixed space points, we move the controllers with certain velocity V in the course of control, i.e., the pinning control now becomes

$$\sum_{i=1}^{M} \varepsilon' \delta(\vec{r} - \vec{r}_i(t))(\vec{u} - \vec{u}_T), \tag{7}$$

where $\vec{r}_i(t) = \vec{r}_i(0) + \vec{V}t$. In equation (7), the moving control is defined in the frame of flow system. Equivalently, in the moving frame fixed with the controllers, the dynamical equation with the control becomes [34]

$$\begin{aligned}\frac{\partial \vec{u}}{\partial t} + \ \vec{u} \cdot \nabla \vec{u} &= -\nabla p - \vec{V} \cdot \nabla \vec{u} + \frac{1}{Re} \nabla^2 \vec{u} \\ &\quad - \sum_{i=1}^{M} \varepsilon' \delta(\vec{r} - \vec{r}_i)(\vec{u} - \vec{u}_T).\end{aligned} \tag{8}$$

In equation (8) $\vec{r}_i, i = 1, 2, ..., M$, are static. But the target $\vec{u}_T$ should be written in the moving frame. The essential difference between equations (5) and (8) is that the latter contains an additional gradient term. In fact, this kind of gradient force is common in many situations. In the following, we will show that this gradient term is crucial for enhancing the efficiency of controlling flow turbulence in the meantime without increasing the injection energy.

It is emphasized that the control scheme equation (8) can be effectively used for realistic turbulence control applications. First, if flow systems have spatially periodic boundary geometries (i.e., in ring- or torus-like containers), the moving control strategy can be easily performed by circling the controllers. Second, if flow systems contain certain gradient forces (like water flow in a sloping channel, or charged particle flow under a constant electrical field), equation (8) can be realized by static controllers (now the first derivative term of equation (8) appears due to the gradient force rather than the moving of the controllers). These two cases include many important and practical situations of turbulence control. Of course, some assumptions on the technical capacities are needed for realizing the control of equation (8). The most important assumptions are the capability to make instant measurements of the system variables at the pinning points and

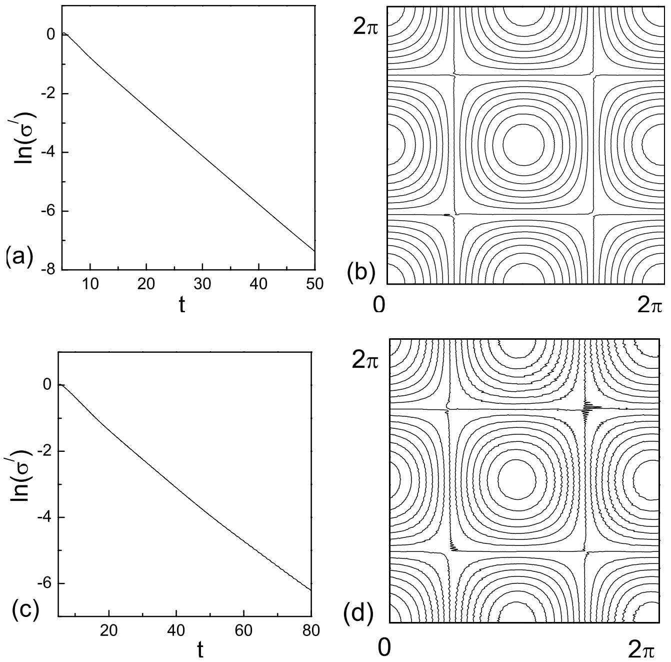

Fig. 3. Controlling flow turbulence with moving controllers. The governing equation is equation (8). $V_x = 1.2$. The other settings are the same as in Figure 2. For (a) and (b), $I_x = 8$, $I_y = 4$. (a) $\ln(\sigma'(t))$ vs t. (b) vorticity contour at $t = 50$. For (c) and (d), $I_x = 8$, $I_y = 8$. (c) $\ln(\sigma'(t))$ vs t. (d) vorticity contour at $t = 80$.

to perform feedback injections responding to the measured data. These are actually the general assumptions for any feedback control. Another technical requirement for experimental realizations is to design moving frame of the controllers with constant (or slightly fluctuated) velocity. This is not always easy, but is experimentally realizable at least, if the flow systems are confined in ring-like containers.

To test the control effect of pinning control with moving controllers, we carried out numerical simulations to equation (8) as we did to obtain results in Figure 2. For simplicity, the controllers are only moving in the x direction throughout this paper, i.e.,

$$\vec{V} = (V_x = 1.2, \quad V_y = 0). \tag{9}$$

Comparing Figure 3 with Figure 2, immediately we can find two improvements using the control strategy with moving controllers. First, with the same M the moving control can suppress flow turbulence significantly faster than the static control. Second, the moving control can successfully suppress turbulence with certain number of controllers M, with which the static control fails. These results do not depend on the target. In our simulations, qualitatively similar results have been obtained when target $\vec{u}_{T1}$ is considered.

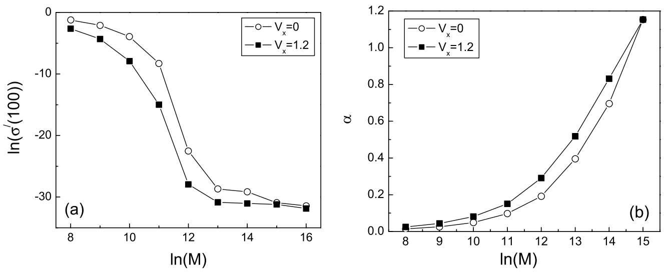

Fig. 4. Characterizing the control efficiency with moving controllers. (a) $\ln(\sigma'(100))$ vs. $\ln(M)$. (b) α vs. $\ln(M)$. See equation (12) for the definitions of $\sigma'(t)$ and α.

4 Analysis of efficiency of moving control

The comparison of Figures 2 and 3 gives a qualitative impression on the effectiveness of the moving control. In this section, we present quantitative characterization on this control strategy. In order to measure the distance between the turbulence system and the target, we can consider the deviation between the vorticity field of the turbulence $\omega(x, y, t)$ and that of the target $\omega_T(x, y, t)$ as

$$\Delta\omega = \omega(x, y, t) - \omega_T(x, y, t), \tag{10}$$

where $\omega(x, y, t) = v_x - u_y$ and the subscript denotes the derivative over the corresponding space variable. We further define the variance of $\Delta\omega$, i.e.,

$$\sigma(t) = \left\{ \frac{1}{(256)^2} \sum_{i=1,j=1}^{256} [\triangle\omega(i, j, t)]^2 \right\}^{1/2} \tag{11}$$

as the quantitative measure of the control efficiency.

In Figure 4a we plot $\ln(\sigma'(100))$ against $\ln(M)$ for both static and moving control with the target $\vec{u}_{T1}$ used. $\sigma'(t)$ is defined as $\sigma(t)/\sigma(0)$ and it decays exponentially as

$$\sigma'(t) \sim e^{-\alpha t}. \tag{12}$$

With fixed ε, the exponent α depends on M and V. In Figure 4b we plot α versus ln$M$ for the same target $\vec{u}_{T1}$. From Figure 4, it can be found that with moving controllers the local feedback control entrains the system to the target faster than that with the static controllers. We emphasize that this result is independent of the target. Similar results have been obtained when target $\vec{u}_{T2}$ is used in simulations. From Figure 4, it can also be seen that the control efficiency between the static and moving control becomes negligible as M is sufficiently large. This is reasonable because with large density of controllers pinning control turns out to be approximately global. In this situation, moving control does not make significant difference from the static one.

Another important quantity for controlling flow turbulence is transient time needed for turbulence suppression. Here, we define the transient time τ as the time interval between the start of control and the moment after which

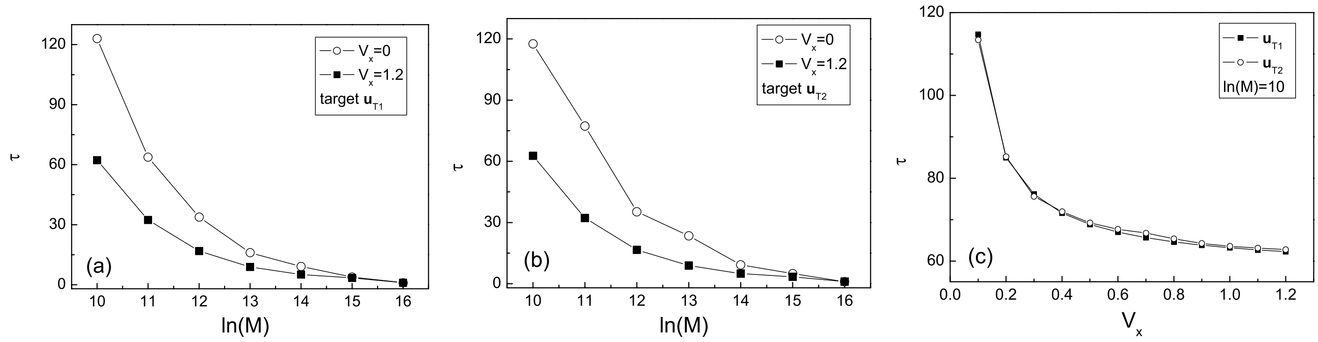

Fig. 5. Comparison between the transient time with static control and that with moving control for different targets. (a) τ vs. $\ln(M)$ for target $\vec{u}_{T1}$. (b) τ vs. $\ln(M)$ for target $\vec{u}_{T2}$. (c) τ vs. V_x. For $V_x = 0$, the static control fails to control turbulence in this situation.

$\sigma'(t)$ becomes smaller than a threshold 5×10^{-3}. In Figures 5a and 5b, we plot τ vs. $\ln M$ for both static and moving control. It is seen that with moving controllers of velocity $V_x = 1.2$ flow turbulence can be suppressed approximately two times faster than that with the static control, provided the same number of controllers M and the same control strength ε. Apparently, this improvement of control speed is due to the gradient term in equation (8). Finally, in order to find how the speed of the moving controllers affect the effectiveness of turbulence control, we study how τ changes with V_x while keeping $M = 32 \times 32$ ($I_x = 8$, and $I_y = 8$) fixed. The results are shown in Figure 5c. It is found that the transient time decreases monotonically when the speed of the moving controllers increases.

The results in the present study can be heuristically understood. With static control, the injected feedback signals first drive the local regions where the control signals are added to the target state. Then the control effect propagates to the regions where no direct control signals are injected. In this process, the spatial correlation of the dynamical system plays an important role. This space correlation length characterizes the effective control area of each controller. Therefore, for static control, it requires sufficiently high density of controllers to achieve successful control. The threshold of the density of the controllers is roughly determined by the requirement that the average distance between two neighbor controllers should be smaller than the space correlation length of the unperturbed system [27]. In the case of moving controllers, the driving signals can inject to all the spatial regions of the paths along which the controllers move. Roughly speaking, the control effect of a moving controller (with moving velocity V) at a space point x and a time moment t can be maintained and supported by the control effect of the successive controllers at late time $t+\frac{\Delta l}{V}$ (Δl is the distance of two neighbor controllers). And this mechanism effectively increases the effective control area of each controller. Therefore, the same or even better control effect can be achieved with smaller number of moving controllers. Nevertheless, the moving control should pay price for this advantage. Unlike the static control, the moving control does not inject feedback signals continuously to any fixed space regions, instead it injects control signals into the regions along the paths of controllers sporadically. In order to make successful sporadic control, time interval between two successive injections in a spatial region should be smaller than the temporal correlation length of the local dynamics. This explains the observation that larger velocity V_x, i.e., smaller time interval between sporadic injections, has better control effect as shown in Figure 5c.

5 Concluding remarks

To conclude, in the present study we suggest to control flow turbulence described by the two-dimensional NSE with moving controllers. Theoretically, this method is equivalent to adding an additional gradient coupling term in the governing equation of velocity field. Numerical experiments demonstrate that the proposed control strategy can enhance the control efficiency while without increasing the number of controllers and the energy of injected signals. Recently, many experimental works have achieved spatiotemporal chaos control with various dynamical control methods [28,32,46,47]. We hope the present theoretical work may attract attention from experimentalists on the flow turbulence control applications.

In this work we analyse flow turbulence control only for a specific Reynolds number $Re = 5000$, and for the simplest periodic boundary condition. The advantages of moving controllers strategy in enhancing efficiency of flow turbulence control exist for different Reynolds numbers. Moreover, we expect that these advantages may exist for more general boundaries, in particular, the boundaries with complex geometry and obstacles. These more complicated problems will be discussed in our future works.

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